SOLUTION OF PROBLEM 89. (*SEE PAGE* 195, *VOL. II.*)

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LET x and 0 be the coordinates of the lower sphere, x' and y' those of the upper, θ and θ' the amounts of rotation, and φ the angle the line joining their centres makes with the horizon, and for brevity put $h = R + r$.

The expression for the living force is

$$T = \frac{m}{2}\left[\frac{dx^2}{dt^2} + \frac{2}{5}R^2\frac{d\theta^2}{dt^2}\right] + \frac{m'}{2}\left[\frac{dx'^2}{dt^2} + \frac{dy'^2}{dt^2} + \frac{2}{5}r^2\frac{d\theta'^2}{dt^2}\right],$$

and the potential is $\qquad \Omega = -m'gy'.$

According to the frictional conditions, the variables x, x', y', θ and θ' satisfy the following equations,

$$\left.\begin{aligned} R\theta - x &= 0, \\ r\theta' + x + h\tan^{-1}\frac{y'}{x'-x} &= 0, \\ \sqrt{[(x'-x)^2 + y'^2]} - h &= 0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

With Lagrange's method of multipliers, if we denote these equations respectively by $L = 0$, $M = 0$, $N = 0$, and the multipliers of their differentials by λ, μ, ν, and take ξ to represent any one of the 5 variables x, x', y', θ, θ', the general differential equation of the problem is

$$\frac{d}{dt}\cdot\frac{dT}{d.\frac{d\xi}{dt}} - \frac{dT}{d\xi} = \frac{d\Omega}{d\xi} + \lambda\frac{dL}{d\xi} + \mu\frac{dM}{d\xi} + \nu\frac{dN}{d\xi}.$$

Applying this in succession to each of the 5 variables and writing for simplicity φ for $\tan^{-1}\frac{y'}{x'-x}$, we get

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -\lambda + \mu[1 + \sin\varphi] - \nu\cos\varphi, \\ m'\frac{d^2x'}{dt^2} &= -\mu\sin\varphi + \nu\cos\varphi, \\ m'\frac{d^2y'}{dt^2} &= -m'g + \mu\cos\varphi + \nu\sin\varphi, \\ \frac{2}{5}mR^2\frac{d^2\theta}{dt^2} &= \lambda R, \\ \frac{2}{5}m'r^2\frac{d^2\theta'}{dt^2} &= \mu r. \end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

Adding the first and second of (2)

$$\frac{d^2(mx + m'x')}{dt^2} = \mu - \lambda.$$

The two first of (1) and the two last of (2) give

$$\lambda = \frac{2}{5} mR \frac{d^2\theta}{dt^2} = \frac{2}{5} m \frac{d^2x}{dt^2},$$

$$\mu = \frac{2}{5} m\, r\, \frac{d^2\theta'}{dt^2} = -\frac{2}{5} m' \left[\frac{d^2x}{dt^2} + h\frac{d^2\varphi}{dt^2}\right].$$

Substituting these values for λ and μ in the last equation,

$$\frac{d^2(mx + m'x')}{dt^2} = -\frac{2}{5}\left(m + m'\right)\frac{d^2x}{dt^2} - \frac{2}{5} m'h \frac{d^2\varphi}{dt^2}. \quad \ldots\ldots \quad (3)$$

Integrating once and eliminating x'

$$\frac{7}{5}\left(m + m'\right)\frac{dx}{dt} + m'h\left(\frac{2}{5} - \sin\varphi\right)\frac{d\varphi}{dt} = 0,$$

where the constant is zero because the spheres are supposed to set out together from a state of rest. As $d\varphi \div dt$, in general, is negative, (φ can always be supposed in the first quadrant), it is evident from this equation, that if $\sin\varphi > \frac{2}{5}$, the lower sphere will move horizontally towards the side on which the upper sphere is; but if $\sin\varphi < \frac{2}{5}$, in the opposite direction.

Integrating (3) twice

$$(7m + 2m')x + 5m'x' + 2m'h\varphi = \text{a constant}.$$

Eliminating x and φ from this by substituting their values in terms of x' and y', we get as the equation of the path of the centre of the upper sphere

$$7\left(m + m'\right)\left[x' - \sqrt{(h^2 - y'^2)}\right] + m'\left[2h\sin^{-1}\frac{y'}{h} + 5\sqrt{(h^2 - y'^2)}\right] = \text{a constant}.$$

As ν denotes the pressure of the upper on lower sphere, the spheres will separate when $\nu = 0$. Now if we eliminate μ between the second and third of (2), we see that $\nu = 0$ is equivalent to

$$\frac{d^2x'}{dt^2}\cos\varphi + \left(\frac{d^2y'}{dt^2} + g\right)\sin\varphi = 0.$$

And if we eliminate x' and y' from this by means of their values in terms of x and φ we get

$$\frac{d^2x}{dt^2}\cos\varphi + g\sin\varphi - h\frac{d\varphi^2}{dt^2} = 0.$$

By eliminating second derivatives this becomes

$$49m\left[\frac{h}{g}\frac{d\varphi^2}{dt^2} - \sin\varphi\right] + 10m'(1 + \sin\varphi)^2\left[\frac{h}{g}\frac{d\varphi^2}{dt^2} - 1\right] = 0,$$

which, by substituting the value of $d\varphi^2 \div dt^2$, becomes

$$70(m + m')[49m + 10m' + 20m'\sin\varphi + 10m'\sin^2\varphi][\sin\beta - \sin\varphi]$$
$$-[10m' + (49m + 20m')\sin\varphi + 10m'\sin^2\varphi][49m + 45m' + 20m'\sin\varphi$$
$$-25m'\sin^2\varphi] = 0.$$